

A geometric proof of Bourgain's L^2 estimate of maximal operators along analytic vector fields

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Abstract

Bourgain [3] proved that the maximal operator associated to an analytic vector field is bounded on L^2 . In the present paper, we give a geometric proof of Bourgain's result by using the tools developed by Lacey and Li in [6] and [7].

1 Statement of the main result

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and $v : \Omega' \rightarrow S^1$ be a unit vector field defined on a neighbourhood Ω' of the closure $\bar{\Omega}$ of Ω . For a fixed small positive number $\epsilon_0 > 0$, define the maximal operator associated to the vector field v truncated at the scale ϵ_0 by

$$M_{v, \epsilon_0} f(x) := \sup_{\epsilon < \epsilon_0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x + tv(x))| dt. \quad (1.1)$$

In [3], Bourgain proved that

Theorem 1.1 ([3]). *Let v be real analytic on Ω' . Then for $\epsilon_0 > 0$ chosen small enough, M_{v, ϵ_0} is bounded on $L^2(\Omega)$.*

Remark 1.2. *The L^p bounds ($\forall p > 1$) for (1.1) and its singular integral variant*

$$H_{v, \epsilon_0} f(x) := \int_{-\epsilon_0}^{\epsilon_0} f(x + tv(x)) \frac{dt}{t} \quad (1.2)$$

were obtained by Stein and Street [8] via a very different method. See also [4] for results concerning certain smooth vector fields.

To prove Theorem 1.1, Bourgain reduced the analyticity assumption on the vector field to the following geometric one: for $x \in \Omega$ and t small enough, define the function

$$\omega_x(t) = |\det[v(x + tv(x)), v(x)]|. \quad (1.3)$$

We assume that

$$\left| \{t \in [-\epsilon, \epsilon] : \omega_x(t) < \tau \sup_{-\epsilon \leq s \leq \epsilon} \omega_x(s)\} \right| \leq C_0 \tau^{c_0} \epsilon, \quad (1.4)$$

for all $0 < \tau < 1, 0 < \epsilon \leq \epsilon_0$, where $0 < c_0, C_0 < \infty$ are constants independent of the point $x \in \Omega$.

It is shown in [3] that Theorem 1.1 can be reduced to the following

Theorem 1.3 ([3]). *If v is C^1 and satisfies the condition (1.4), then M_{v, ϵ_0} is bounded on $L^2(\Omega)$.*

Bourgain's proof for Theorem 1.3 is not entirely geometric, especially in the key Lemma 3.28, where he used polar coordinates and applied Schur's lemma to get the desired L^2 bounds.

The goal of this paper is to give a (relatively) geometric proof of Theorem 1.3. The idea is to use the time-frequency decomposition initiated by Lacey and Li in the setting of the Hilbert transform along vector fields in [6] and [7]. This decomposition was further developed by Bateman [1], Bateman and Thiele [2]. We refer to [5] for the detailed description of the progress that has been made for Hilbert transforms and maximal operators along vector fields.

This paper is free of time-frequency analysis techniques. Other than certain geometric lemmas (in the following Section 4), the tools that we will be using are only the elementary ones, for example Plancherel's identity, the Cauchy-Schwartz inequality, Minkowski's inequality and so on.

Notations: Throughout this paper, we will write $x \ll y$ to mean that $x \leq y/10$, $x \lesssim y$ to mean that there exists a universal constant C s.t. $x \leq Cy$, and $x \sim y$ to mean that $x \lesssim y$ and $y \lesssim x$. $\mathbb{1}_E$ will always denote the characteristic function of the set E .

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2 Reduction to a smooth cut-off

By a renormalization, we assume further that $\|v\|_{C^1} \leq 1$, and $\Omega = B_{\epsilon_0}(0)$, which is the ball of radius $\epsilon_0 \ll 1$ centered at the origin. Moreover, as we are only concerned with the truncated maximal operator, we can w.l.o.g. assume that the vector field is periodic in both horizontal and vertical directions with each periodicity being $3 \cdot \epsilon_0$, and that the vector field always points in the two-ended cone which forms an angle less than $\pi/10$ with the horizontal axis. In the following, we will denote this cone by Γ_0 .

Choose $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ to be a proper smooth bump function such that the support of $\hat{\alpha}$ lies on $[-1, 1]$. For $0 < \epsilon \leq \epsilon_0$, define

$$A_\epsilon f(x) := \int_{\mathbb{R}} f(x + \epsilon v(x)) \alpha(t) dt. \quad (2.1)$$

It is not difficult to see that to bound the maximal function $M_{v, \epsilon_0} f$, it is essentially equivalent to bound

$$\sup_{j \in \mathbb{N}, 2^{-j} \leq \epsilon_0} |A_{2^{-j}} f(x)|, \quad (2.2)$$

which will still be called $M_{v,\epsilon_0}f$. Moreover we will write A^j to stand for $A_{2^{-j}}$ for the sake of simplicity. Hence in the rest of the paper, we will focus on the following operator

$$M_{v,\epsilon_0}f(x) := \sup_{j \in \mathbb{N}, 2^{-j} \leq \epsilon_0} |A^j f(x)|, \quad (2.3)$$

and prove that it is bounded on L^2 .

3 Bourgain's high-low frequency decomposition

We linearize the maximal operator in (2.3): Take a measurable function $J : \mathbb{R}^2 \rightarrow \mathbb{N}$ such that

$$M_{v,\epsilon_0}f(x) \sim \left| A^{J(x)} f(x) \right|. \quad (3.1)$$

For a point $x \in \Omega = B_{\epsilon_0}(0)$, let $R_{x,j}$ be the rectangle with center x , orientation $v(x)$, length 2^{-j} in direction $v(x)$ and width

$$\delta(R_{x,\epsilon}) = 2^{-j} \cdot \sup_{|t| < 2^{-j}} \omega_x(t). \quad (3.2)$$

Especially we denote

$$\delta(x) := 2^{-J(x)} \cdot \sup_{|t| < 2^{-J(x)}} \omega_x(t). \quad (3.3)$$

Choose a measurable function $K : \mathbb{R}^2 \rightarrow \mathbb{N}$ such that

$$\delta(x) \sim 2^{-K(x)}, \forall x \in \mathbb{R}^2. \quad (3.4)$$

Do a Littlewood-Paley decomposition for the function f , and write

$$f = \sum_{k \in \mathbb{Z}} P_k f. \quad (3.5)$$

This turns our linearised maximal operator into

$$\sum_{k \in \mathbb{Z}} A^{J(x)} P_k f(x). \quad (3.6)$$

Bourgain's idea is to split the function f into two parts, the high frequency part and the low frequency part, in the following way:

$$\sum_{k \in \mathbb{Z}} A^{J(x)} P_k f(x) = \sum_{k \in \mathbb{Z}, k \geq K(x)} A^{J(x)} P_k f(x) + \sum_{k \in \mathbb{Z}, k < K(x)} A^{J(x)} P_k f(x). \quad (3.7)$$

For the latter part, i.e. the low frequency part, Bourgain's proof is already geometric, see Lemma 4.12 and Lemma 5.7 in [3]. Hence the main task for us is to bound the former part, i.e. the high frequency part, by a geometric argument.

Remark 3.1. *The estimate of the above high frequency part is done in Lemma 3.28 in [3] via analytic methods.*

We proceed with the estimate of the high frequency part: First we write

$$\sum_{k \in \mathbb{Z}, k \geq K(x)} A^{J(x)} P_k f(x) = \sum_{l \in \mathbb{N}_0} A^{J(x)} P_{K(x)+l} f(x). \quad (3.8)$$

Then by the triangle inequality, it suffices to prove that

$$\|A^{J(x)} P_{K(x)+l} f(x)\|_2 \lesssim 2^{-\mu l} \|f\|_2, \quad (3.9)$$

for some $\mu > 0$, with a constant being independent of $l \in \mathbb{N}$.

Notice that the above estimate is still of a maximal type, and we want to get rid of the linearization by replacing the l^∞ norm by an l^2 norm. To do this, we need to introduce several notations. For $j, k \in \mathbb{N}$, define

$$\Omega_{j,k} := \{x \in \Omega | 2^{-j} \cdot \sup_{|t| < 2^{-j}} \omega_x(t) \sim 2^{-k}\}. \quad (3.10)$$

For a real analytic vector field, either the integral curves are straight lines, or for each $j \in \mathbb{N}$, the complement of the set $\cup_k \Omega_{j,k}$ has measure zero. Hence it is no restriction to assume for each $j \in \mathbb{N}$ that

$$\Omega = \bigcup_{k \in \mathbb{N}} \Omega_{j,k}. \quad (3.11)$$

It is also clear that for a fixed $k \in \mathbb{N}$, the $\Omega_{j,k}$ for different j are essentially disjoint. Hence for a fixed x ,

$$|A^{J(x)} P_{K(x)+l} f(x)| \lesssim \sup_{j \in \mathbb{N}} \left(\sum_{k \in \mathbb{N}} |A^j P_{k+l} f|^2 \mathbb{1}_{\Omega_{j,k}} \right)^{1/2}. \quad (3.12)$$

We replace the sup norm by the l^2 norm to obtain

$$|A^{J(x)} P_{K(x)+l} f(x)| \lesssim \left(\sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} |A^j P_{k+l} f|^2 \mathbb{1}_{\Omega_{j,k}} \right)^{1/2}. \quad (3.13)$$

Taking the L^2 norm of (3.13), we obtain

$$\sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} \int |A^j P_{k+l} f|^2 \mathbb{1}_{\Omega_{j,k}}. \quad (3.14)$$

Recall that $R_{x,j}$ denotes the rectangle with center x , length 2^{-j} and width

$$\delta(R_{x,j}) := 2^{-j} \sup_{|t| < 2^{-j}} \omega_x(t). \quad (3.15)$$

In the following we will cover $\Omega_{j,k}$ by rectangles $\{R_m = R_{x_m,j}\}_{m \in \mathbb{N}}$ satisfying the following two conditions

- (i) $\delta(R_m) = 2^{-k}$;
 - (ii) the center of R_m does not belong to $R_1 \cup \dots \cup R_{m-1}$.
- (3.16)

Hence for fixed j and k ,

$$\int |A^j P_{k+l} f|^2 \mathbb{1}_{\Omega_{j,k}} \lesssim \sum_{m \in \mathbb{N}} \int |A^j P_{k+l} f|^2 \mathbb{1}_{R_m}. \quad (3.17)$$

Indeed, the above covering of $\Omega_{j,k}$ is a “valid” covering, i.e. a covering without much overlapping. To be precise, if we define

$$\Omega'_{j,k} := \bigcup_{x \in \Omega_{j,k}} 2 \cdot R_{x,j}, \quad (3.18)$$

then it has been proved by Bourgain in [3] (see the following Lemma 4.5) that

$$\left\| \sum_j \mathbb{1}_{\Omega'_{j,k}} \right\|_{\infty} \lesssim 1. \quad (3.19)$$

In the following, when estimating the right hand side of (3.17), we will need several other geometric properties like (3.19). Hence we organize all them together in the next section.

4 Geometric properties of rectangles

In this section we collect several geometric lemmas that will play crucial roles in the forthcoming calculation. Most of these lemmas have already been proven by Bourgain [3]. Here we still include them for the sake of completeness. Moreover, out of certain technical reasons, we will still need several variants of these geometric lemmas, which are the following Lemma 4.3 and 4.6.

Lemma 4.1 (Lemma 4.1 in [3]). *Let x' be in the rectangle $R_{x,j}$, then*

$$\delta(R_{x,j}) \sim \delta(R_{x',j}), \quad (4.1)$$

and $R_{x,j}$ is contained in a multiple of $R_{x',j}$ and vice versa.

Lemma 4.2 (Lemma 4.6 in [3]). *Assume*

$$2 \cdot R_{x,j} \cap 2 \cdot R_{x',j'} \neq \emptyset, \quad (4.2)$$

and $2^{-j'} \lesssim 2^{-j}$. Then

$$R_{x',j'} \subset 4 \cdot R_{x,j} \quad (4.3)$$

and

$$\frac{\delta(R_{x,j})}{2^{-j}} \gtrsim \frac{\delta(R_{x',j'})}{2^{-j'}}. \quad (4.4)$$

Namely the larger rectangle has the larger eccentricity.

Lemma 4.3. *Assume that*

$$R_{x,j} \cap R_{x',j+j_0} \neq \emptyset, \quad (4.5)$$

for some $j_0 \in \mathbb{N}_0$. Then there exists a constant $a_0 > 1$ such that

$$\frac{\delta(R_{x,j})}{2^{-j}} \lesssim (2^{j_0})^{a_0} \cdot \frac{\delta(R_{x',j+j_0})}{2^{-j-j_0}}. \quad (4.6)$$

Remark 4.4. Compared with Lemma 4.2, this lemma says that the growth of the eccentricity of a rectangle with respect to its length can only be polynomial.

Proof of Lemma 4.3: This follows easily from Lemma 4.1 and the following doubling estimate (3.20) in [3]

$$\frac{\delta(R_{x,j})}{2^{-j}} \leq C \cdot \frac{\delta(R_{x',j+1})}{2^{-j-1}}, \quad (4.7)$$

for some constant $C > 0$. \square

Lemma 4.5 (Lemma 4.7 in [3]). *Let $\{R_{x_i,j_i}\}_{i \in \mathbb{N}_0}$ be a sequence of rectangles and $\delta > 0$ such that*

$$\begin{aligned} (i) \quad & \delta(R_{x_i,j_i}) \sim \delta; \\ (ii) \quad & x_{i+1} \text{ does not belong to } R_{x_0,j_0} \cup \dots \cup R_{x_i,j_i}, \forall i. \end{aligned} \quad (4.8)$$

Then

$$\left\| \sum_{i \in \mathbb{N}_0} \mathbb{1}_{2 \cdot R_{x_i,j_i}} \right\|_\infty \lesssim 1. \quad (4.9)$$

We will also need the following generalized version of the above lemma.

Lemma 4.6. *Under the same assumptions as in Lemma 4.5, there exists a constant $b_0 > 0$ such that for any $N \in \mathbb{N}_0$, we have*

$$\left\| \sum_{i \in \mathbb{N}_0} \mathbb{1}_{R_{x_i,j_i}^{p,q}} \right\|_\infty \lesssim (p+q+1)^{b_0}, \forall p, q \in \mathbb{N}, \quad (4.10)$$

where $R_{x_i,j_i}^{p,q}$ is obtained by dilating the length of R_{x_i,j_i} to p times, and the width to q times.

Remark 4.7. The above lemma says that when we enlarge the rectangles R_{x_i,j_i} , their overlapping can only be polynomially growing.

Proof of Lemma 4.6: Denote

$$\tilde{q} := \max\{q, p^{a_0+1}\}, \quad (4.11)$$

where a_0 is the constant in (4.6). For the rectangle $R_{x_i,j_i}^{p,q}$, we further enlarge it to be of width $\tilde{q} \cdot \delta$. Next, we will dilate the length to $\tilde{p}_i \cdot 2^{-j_i}$ such that

$$\sup_{|t| \leq \tilde{p}_i \cdot 2^{-j_i}} \omega_{x_i}(t) \sim \frac{\tilde{q}}{\tilde{p}_i} \cdot \frac{\delta}{2^{-j_i}}. \quad (4.12)$$

By Lemma 4.2, it is not difficult to see that

$$\tilde{p}_i \leq \tilde{q} \lesssim \tilde{p}_i^{a_0+1}, \quad (4.13)$$

uniformly in i .

Our goal now is to show that

$$\left\| \sum_i \mathbb{1}_{R_{x_i,j_i}^{\tilde{p}_i,\tilde{q}}} \right\|_\infty \lesssim \tilde{q}^{b_0}, \quad (4.14)$$

for some b_0 to be determined later. Suppose that the L^∞ norm on the left hand side of the above expression is attained at the point O . Moreover, let \mathcal{R}_O denote the collection of rectangles containing the point O . W.l.o.g. we assume that

$$\mathcal{R}_O = \{R_{x_i, j_i}^{\tilde{p}_i, \tilde{q}}\}_{0 \leq i \leq N}, \quad (4.15)$$

for some $N \in \mathbb{N}_0$. Then (4.14) is equivalent to proving

$$N \lesssim \tilde{q}^{b_0}. \quad (4.16)$$

By the definition of the rectangles $R_{x_i, j_i}^{\tilde{p}_i, \tilde{q}}$ and Lemma 4.1, it is not difficult to see that all the rectangles in \mathcal{R}_O have comparable lengths. Indeed, up to a constant dilation factor, any of these rectangles is contained in another. Hence

$$\bigcup_{0 \leq i \leq N} R_{x_i, j_i}^{\tilde{p}_i, \tilde{q}} \subset 4R_{x_0, j_0}^{\tilde{p}_0, \tilde{q}}. \quad (4.17)$$

Moreover, by the upper bound on \tilde{p}_i in (4.13), we can also obtain that

$$l(R_{x_i, j_i}) \gtrsim l(R_{x_0, j_0}^{\tilde{p}_0, \tilde{q}})/\tilde{q}, \quad (4.18)$$

where for a rectangle R , $l(R)$ is used to denote its length. Hence by the assumption that the center x_i of R_{x_i, j_i} is not contained in

$$R_{x_0, j_0} \cup \dots \cup R_{x_{i-1}, j_{i-1}} \quad (4.19)$$

for all i , we obtain easily the estimate (4.16) for some constant b_0 depending only on a_0 . So far we have finished the proof of Lemma 4.6. \square

5 Estimate on each rectangle

We proceed with the estimate of the right hand side of (3.17). In this section, we will prove an estimate for fixed j, k, l and for a given rectangle R_m . Here R_m is a rectangle of length 2^{-j} and width 2^{-k} .

Recall that we have assumed that the vector field points in the cone Γ_0 , which is the two-ended cone forming an angle less than $\pi/10$ with the horizontal axis. If we denote by P_{Γ_0} the frequency projection operator on the cone Γ_0 , it is not difficult to see that

$$A^j P_{k+l} P_{\Gamma_0} f \equiv 0. \quad (5.1)$$

Hence in the following we will only be concerned with the frequency in the cone Γ_0^c . Moreover, for the sake of simplicity, we will always identify P_{k+l} with $P_{k+l} P_{\Gamma_0^c}$.

5.1 Time-frequency decomposition of the function $P_{k+l} f$

Most of the content in this subsection is taken from Bateman [1]. Here we will make some modifications as we will be dealing with all frequency annuli instead of one single annulus.

Frequency decomposition. For the fixed j, k and l , we will denote

$$\theta := k + l - j. \quad (5.2)$$

Let \mathcal{D}_θ be the collection of the dyadic intervals of length $2^{-\theta}$ contained in $[-20, 20]$. Fix a smooth positive function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ s.t.

$$\beta(x) = 1, \forall |x| \leq 1; \beta(x) = 0, \forall |x| \geq 2. \quad (5.3)$$

Also choose β such that $\sqrt{\beta}$ is a smooth function. For each $\omega \in \mathcal{D}_\theta$, define

$$\beta_\omega(x) = \beta(2^\theta(x - c_\omega)), \quad (5.4)$$

where c_ω denotes the center of the interval ω . Define

$$\beta_\theta(x) = \sum_{\omega \in \mathcal{D}_\theta} \beta_\omega(x). \quad (5.5)$$

Notice that

$$\beta_\theta(x + 2^{-\theta}) = \beta_\theta(x), \forall x \in [-20, 20 - 2^{-\theta}]. \quad (5.6)$$

Define

$$\gamma_\theta = \frac{1}{20} \int_{-10}^{10} \beta_\theta(x + t) dt. \quad (5.7)$$

Because of the above periodicity, we know that γ_θ is constant for $x \in [-10, 10]$, independently of θ . Say $\gamma_\theta(x) = \gamma > 0$, hence

$$\frac{1}{\gamma} \cdot \gamma_\theta(x) \mathbb{1}_{[-10, 10]}(x) = \mathbb{1}_{[-10, 10]}(x). \quad (5.8)$$

Define another multiplier $\tilde{\beta} : \mathbb{R} \rightarrow \mathbb{R}$ with support in $[\frac{1}{2}, \frac{5}{2}]$ and $\tilde{\beta}(x) = 1$ for $x \in [1, 2]$. We define the corresponding multipliers on \mathbb{R}^2 :

$$\begin{aligned} \hat{m}_{k+l, \omega}(\xi, \eta) &= \tilde{\beta}(2^{-k-l}\eta) \beta_\omega\left(\frac{\xi}{\eta}\right) \\ \hat{m}_{k+l, \theta, t}(\xi, \eta) &= \tilde{\beta}(2^{-k-l}\eta) \beta_\theta\left(t + \frac{\xi}{\eta}\right) \\ \hat{m}_{k+l, \theta}(\xi, \eta) &= \tilde{\beta}(2^{-k-l}\eta) \gamma_\theta\left(\frac{\xi}{\eta}\right) \end{aligned}$$

Using the above multipliers, we obtain

$$A^j P_{k+l} f = A^j(m_{k+l, \theta} * f) = \frac{1}{20} \int_{-10}^{10} A^j(m_{k+l, \theta, t} * f) dt. \quad (5.9)$$

Hence it suffices to prove a uniform bound on $t \in [-10, 10]$. W.l.o.g. we will just consider the case $t = 0$, which is

$$A^j(m_{k+l, \theta, 0} * f) = \sum_{\omega \in \mathcal{D}_\theta} A^j(m_{k+l, \omega} * f). \quad (5.10)$$

Space (Time) decomposition: For $\omega \in \mathcal{D}_\theta$, let $\mathcal{U}_{k+l, \omega}$ be a partition of \mathbb{R}^2 by rectangles of width 2^{-k-l} and length 2^{-j} , whose long side have slope $-c_\omega$ with c_ω denoting the center of the interval ω . If $s \in \mathcal{U}_{k+l, \omega}$, we will write $\omega_s := \omega$.

Definition 5.1. For a rectangle $R \subset \mathbb{R}^2$ of slope less than one, with $l(R)$ its length, $w(R)$ its width, we define its uncertainty interval $EX(R)$ to be the interval of width $w(R)/l(R)$ and centered at $\text{slope}(R)$.

Remark 5.2. For a tile $s \in \mathcal{U}_{k+l,\omega}$, we have that $EX(s) = -\omega$.

An element of $\mathcal{U}_{k+l,\omega}$ for some $\omega \in \mathcal{D}_\theta$ is called a “tile”. Define $\varphi_{k+l,\omega}$ such that

$$|\hat{\varphi}_{k+l,\omega}|^2 = \hat{m}_{k+l,\omega}, \quad (5.11)$$

then $\varphi_{k+l,\omega}$ is smooth by our assumption on β mentioned above.

For a tile $s \in \mathcal{U}_{k+l,\omega}$, define

$$\varphi_s(p) := \sqrt{|s|} \varphi_{k+l,\omega}(p - c_s), \quad (5.12)$$

where c_s is the center of s . Notice that

$$\|\varphi_s\|_2^2 = \int_{\mathbb{R}^2} |s| \varphi_{k+l,\omega}^2 = |s| \int_{\mathbb{R}^2} \hat{m}_{k+l,\omega} = 1, \quad (5.13)$$

i.e. φ_s is L^2 normalized.

The construction of the tiles above by the uncertainty principle is to localize functions further in space, for this purpose we need

Lemma 5.3. ([1]) Under the above notations, for the frequency localised function $f * m_{k+l,\omega}$, we have

$$f * m_{k+l,\omega}(x) = \lim_{N \rightarrow \infty} \frac{1}{4N^2} \int_{[-N,N]^2} \sum_{s \in \mathcal{U}_{k+l,\omega}} \langle f, \varphi_s(p + \cdot) \rangle \varphi_s(p + x) dp. \quad (5.14)$$

The above lemma allows us to pass the expression in (5.10) to the model sum

$$\sum_{\omega \in \mathcal{D}_\theta} \sum_{s \in \mathcal{U}_{k+l,\omega}} \langle f, \varphi_s \rangle A^j \varphi_s.$$

Definition 5.4. For a unit vector $v = (v_1, v_2) \in \mathbb{R}^2$, for an interval $I \subset \mathbb{R}$, we say that $v \in I$ if the slope of the line that is perpendicular to v lies in I .

Lemma 5.5. ([6], [1]) we have that $A^j \varphi_s(x) = 0$ unless $v(x) \in \omega_s$.

To have this lemma is the main reason of replacing the strict cut-off by a smooth cut-off in Section 2. The proof of Lemma 5.5 is simply by the Plancherel theorem. From this lemma we know that in order for the output $A^j \varphi_s$ not to vanish at a point x , the vector field at x has to point roughly to the direction of the long side of the tile s .

5.2 Estimate on each rectangle by ignoring the tails of the wavelet functions

After the above preparation, we turn to the estimate of each term on the right hand side of (3.17), which is

$$\int |A^j P_{k+l} f|^2 \mathbb{1}_{R_m} = \int \left| \sum_{\omega \in \mathcal{D}_\theta} \sum_{s \in \mathcal{U}_{k+l,\omega}} \langle f, \varphi_s \rangle A^j \varphi_s \right|^2 \mathbb{1}_{R_m}. \quad (5.15)$$

By Lemma 5.5, we have that the right hand side of (5.15) is equal to

$$\sum_{\omega \in \mathcal{D}_\theta} \int \left| \sum_{s \in \mathcal{U}_{k+l, \omega}} \langle f, \varphi_s \rangle A^j \varphi_s \right|^2 \mathbb{1}_{R_m}. \quad (5.16)$$

In this subsection, we will only show the ideas of how to bound the above term, or in another word, we will ignore the tails of the wavelet functions and the function α in the definition of A^j in (2.1), and always assume that they have compact support in both space and frequency.

Under the above simplification, the expression in (5.16) becomes

$$\sum_{\omega \in \mathcal{D}_\theta} \sum_{s \in \mathcal{U}_{k+l, \omega}} |\langle f, \varphi_s \rangle|^2 \int |A^j \varphi_s|^2 \mathbb{1}_{R_m}. \quad (5.17)$$

Take a point $x \in R_m$, for a tile $s \in \mathcal{U}_{k+l, \omega}$ for some $\omega \in \mathcal{D}_\theta$, we observe that in order for $A^j \varphi_s(x)$ not to vanish, we must have $\omega \subset 3 \cdot EX(R_m)$ as by Lemma 4.1 we know that $v(x) \in 2 \cdot EX(R_m)$ for any $x \in R_m$. This, together with the fact that both R_m and s have length 2^{-j} , implies that

$$s \subset 4 \cdot R_m, \quad (5.18)$$

for those tiles s such that $A^j \varphi_s$ is not identically zero.

Claim 5.6. *There exists $\mu > 0$ such that*

$$\int |A^j \varphi_s|^2 \mathbb{1}_{R_m} \lesssim 2^{-\mu l}, \quad (5.19)$$

with the constant being independent of s .

By the above claim, the expression in (5.17) can be further bounded by

$$\sum_{\omega \in \mathcal{D}_\theta} \sum_{s \in \mathcal{U}_{k+l, \omega}, s \subset 4 \cdot R_m} 2^{-\mu l} \cdot |\langle f, \varphi_s \rangle|^2 \lesssim 2^{-\mu l} \|\mathbb{1}_{R_m} \cdot P_{k+l} f\|_2^2. \quad (5.20)$$

The next step is to sum over m, j and k :

$$\begin{aligned} \sum_{j,k} \sum_m \|\mathbb{1}_{R_m} \cdot P_{k+l} f\|_2^2 &\lesssim 2^{-\mu l} \sum_{j,k} \|\mathbb{1}_{\Omega'_{j,k}} \cdot P_{k+l} f\|_2^2 \\ &\lesssim 2^{-\mu l} \sum_k \|P_{k+l} f\|_2^2 \lesssim 2^{-\mu l} \|f\|_2^2, \end{aligned} \quad (5.21)$$

where we have used the disjointness property (3.19). Hence for the model problem, what remains is

“Proof” of Claim 5.6: We can w.l.o.g. assume that there exists a point $x_0 \in s$ such that

$$v(x_0) \in \omega_s, \quad (5.22)$$

as otherwise $A^j \varphi_s$ will be identically zero. By a proper translation and rotation, we can assume that $x_0 = (0, 0)$ and $v(x_0) = (1, 0)$.

Now we look at the directions of the vector field at the points lying on the line segment

$$\{(x_1, x_2) : x_2 = 0\} \cap s. \quad (5.23)$$

By the assumption on the rectangle R_m we know that

$$\sup_{|t| \leq 2^{-j}} w_{x_0}(t) = \sup_{|t| \leq 2^{-j}} |\det[v(x_0 + tv(x_0)), v(x_0)]| \sim 2^{-k+j}. \quad (5.24)$$

Notice that $|\omega_s| = 2^{-k-l+j}$, hence in order for $A^j \varphi_s$ not to vanish at a point $x \in s \cap \{(x_1, x_2) : x_2 = 0\}$, we must have

$$|\det[v(x), v(x_0)]| = w_{x_0}(x \cdot v(x_0)) \lesssim 2^{-k-l+j}. \quad (5.25)$$

By taking $\tau = 2^{-l}$ in the condition (1.4) we obtain

$$\left| t \in [-2^{-j}, 2^{-j}] : w_{x_0}(t) < 2^{-l} \sup_{|t| \leq 2^{-j}} w_{x_0}(t) \right| \leq C_0 2^{-c_0 l} \cdot 2^{-j}, \quad (5.26)$$

which further implies that

$$|\{(x_1, 0) \in s : A^j \varphi_s(x_1, 0) \neq 0\}| \lesssim 2^{-c_0 l} \cdot 2^{-j}. \quad (5.27)$$

So far we have proved that on one line segment, the non-vanishing output $A^j \varphi_s$ has relatively small measure. In the next, we want to show that this indeed holds true for all the points in the tile s , namely

$$|\{x \in s : A^j \varphi_s(x) \neq 0\}| \lesssim 2^{-c_0 l} |s|. \quad (5.28)$$

This, combined with the trivial estimate

$$\|A^j \varphi_s\|_\infty \lesssim |s|^{-1/2}, \quad (5.29)$$

concludes the proof of Claim 5.6.

We turn to the proof of (5.28): For $|x_2| \leq 2^{-k-l+2}$, consider the line segment

$$L_{x_2} := \{(0, x_2) + t \cdot v(0, x_2) : |t| \leq 2^{-j+2}\}. \quad (5.30)$$

First by the C^1 assumption on the vector field, we know that

$$v(0, x_2) \in 2 \cdot \omega_s, \forall |x_2| \leq 2^{-k-l+2}. \quad (5.31)$$

Then by the same argument as before, we obtain that

$$|\{x \in L_{x_2} : A^j \varphi_s(x) \neq 0\}| \lesssim 2^{-c_0 l} \cdot 2^{-j}, \quad (5.32)$$

for each $|x_2| \leq 2^{-k-l+2}$. Hence by Fubini's theorem (which can be applied due to the C^1 assumption on the vector field), we obtain

$$\begin{aligned} & |\{x \in s : A^j \varphi_s(x) \neq 0\}| \\ &= \int_{-2^{-k-l+2}}^{2^{-k-l+2}} |\{x \in L_{x_2} : A^j \varphi_s(x) \neq 0\}| dx_2 \lesssim 2^{-c_0 l} \cdot 2^{-k-l-j}. \end{aligned} \quad (5.33)$$

Hence we have finished the proof of (5.28).

5.3 The full estimate on each rectangle

In this subsection, we will make the above heuristic argument rigorous, i.e. we will also take care of the tails of the wavelet functions. For fixed j, k, l and m , we want to bound the following

$$\int |A^j P_{k+l} f|^2 \mathbb{1}_{R_m} = \sum_{\omega \in \mathcal{D}_\theta} \int \left| \sum_{s \in \mathcal{U}_{k+l, \omega}} \langle f, \varphi_s \rangle A^j \varphi_s \right|^2 \mathbb{1}_{R_m}. \quad (5.34)$$

For $p, q \in \mathbb{Z}$, we denote by $\vec{R}_m^{p,q}$ the translation of the rectangle R_m by (p, q) units, i.e.

$$\vec{R}_m^{p,q} = R_m + p \cdot 2^{-j} v(x_m) + q \cdot 2^{-k} v^\perp(x_m), \quad (5.35)$$

where x_m denotes the center of R_m and $v(x_m)$ is the value of the vector field at the point x_m which is parallel to the long side of R_m .

Hence for one fixed $\omega \in \theta$, we have

$$\int \left| \sum_{s \in \mathcal{U}_{k+l, \omega}} \langle f, \varphi_s \rangle A^j \varphi_s \right|^2 \mathbb{1}_{R_m} = \int \left| \sum_{p, q \in \mathbb{Z}} \sum_{s \in \mathcal{U}_{k+l, \omega}, s \in \vec{R}_m^{p,q}} \langle f, \varphi_s \rangle A^j \varphi_s \right|^2 \mathbb{1}_{R_m}. \quad (5.36)$$

By Minkowski's inequality, the right hand side of the above display can be bounded by

$$\left(\sum_{p, q \in \mathbb{Z}} \left(\int \left| \sum_{s \in \mathcal{U}_{k+l, \omega}, s \in \vec{R}_m^{p,q}} \langle f, \varphi_s \rangle A^j \varphi_s \right|^2 \mathbb{1}_{R_m} \right)^{1/2} \right)^2. \quad (5.37)$$

Lemma 5.7. *For any large $M \in \mathbb{N}_0$, there exists a constant C_M such that*

$$\int \left| \sum_{s \in \mathcal{U}_{k+l, \omega}, s \in \vec{R}_m^{p,q}} \langle f, \varphi_s \rangle A^j \varphi_s \right|^2 \mathbb{1}_{R_m} \lesssim \frac{C_M \cdot 2^{-\mu l}}{(|p| + |q| + 1)^M} \sum_{s \in \mathcal{U}_{k+l, \omega}, s \in \vec{R}_m^{p,q}} |\langle f, \varphi_s \rangle|^2, \quad (5.38)$$

where μ is the same as the one in Claim 5.6.

We substitute the estimate in Lemma 5.7 into (5.37) to obtain

$$\begin{aligned} & \left(\sum_{p, q \in \mathbb{Z}} \left(\frac{2^{-\mu l}}{(|p| + |q| + 1)^M} \sum_{s \in \mathcal{U}_{k+l, \omega}, s \in \vec{R}_m^{p,q}} |\langle f, \varphi_s \rangle|^2 \right)^{1/2} \right)^2 \\ & \lesssim \sum_{p, q \in \mathbb{Z}} \frac{2^{-\mu l}}{(|p| + |q| + 1)^M} \sum_{s \in \mathcal{U}_{k+l, \omega}, s \in \vec{R}_m^{p,q}} |\langle f, \varphi_s \rangle|^2, \end{aligned} \quad (5.39)$$

where the exact value of M might vary from line to line. Hence we have obtained

$$\int |A^j P_{k+l} f|^2 \mathbb{1}_{R_m} \lesssim \sum_{\omega \in \mathcal{D}_\theta} \sum_{p, q \in \mathbb{Z}} \frac{2^{-\mu l}}{(|p| + |q| + 1)^M} \sum_{s \in \mathcal{U}_{k+l, \omega}, s \in \vec{R}_m^{p,q}} |\langle f, \varphi_s \rangle|^2, \quad (5.40)$$

which is the estimate on one single rectangle that we are aiming at.

Proof of Lemma 5.7: We will only consider the case $p = q = 0$. The decay in p and q in the other case will simply follow from the non-stationary phase method. Hence what we need to prove becomes

$$\int \left| \sum_{s \in \mathcal{U}_{k+l, \omega}, s \subset R_m} \langle f, \varphi_s \rangle A^j \varphi_s \right|^2 \mathbb{1}_{R_m} \lesssim 2^{-\mu l} \sum_{s \in \mathcal{U}_{k+l, \omega}, s \subset R_m} |\langle f, \varphi_s \rangle|^2. \quad (5.41)$$

Recall that each tile s has width 2^{-k-l} , however the rectangle R_m has width 2^{-k} . This suggests that we should do a further partition of R_m into smaller rectangles which will be of the same dimension as s .

We enumerate the tiles $s \subset R_m$ from above to below by $s_1, s_2, \dots, s_{m'} \dots$, where $m' \lesssim 2^l$. Notice that

$$R_m \subset \bigcup_{m'} 2 \cdot s_{m'}. \quad (5.42)$$

Hence

$$\int \left| \sum_{s \in \mathcal{U}_{k+l, \omega}, s \subset R_m} \langle f, \varphi_s \rangle A^j \varphi_s \right|^2 \mathbb{1}_{R_m} \lesssim \sum_{m'} \int \left| \sum_{m''} \langle f, \varphi_{s_{m''}} \rangle A^j \varphi_{s_{m''}} \right|^2 \mathbb{1}_{2 \cdot s_{m'}}. \quad (5.43)$$

By the Cauchy-Schwartz inequality, we bound the right hand side of the above expression by

$$\sum_{m'} \int \sum_{m''} \left| \langle f, \varphi_{s_{m''}} \rangle A^j \varphi_{s_{m''}} \right|^2 (|m' - m''| + 1)^M \mathbb{1}_{2 \cdot s_{m'}}, \quad (5.44)$$

for some large constant M . By the same argument as in the proof of Claim 5.6, we obtain

$$|\{x \in 2 \cdot s_{m'} : A^j \varphi_{s_{m''}} \neq 0\}| \lesssim 2^{-c_0 l} |s_{m'}|. \quad (5.45)$$

This, together with the trivial bound

$$\|A^j \varphi_{s_{m''}}\|_{L^\infty(s_{m'})} \lesssim \frac{|s_{m''}|^{1/2}}{(|m' - m''| + 1)^{2M}}, \quad (5.46)$$

implies that

$$\int \left| \langle f, \varphi_{s_{m''}} \rangle A^j \varphi_{s_{m''}} \right|^2 \mathbb{1}_{2 \cdot s_{m'}} \lesssim \frac{2^{-c_0 l}}{(|m' - m''| + 1)^{2M}} |\langle f, \varphi_{s_{m''}} \rangle|^2. \quad (5.47)$$

We substitute the above estimate into (5.44) to obtain

$$\sum_{m'} \sum_{m''} \frac{2^{-c_0 l}}{(|m' - m''| + 1)^M} |\langle f, \varphi_{s_{m''}} \rangle|^2 \lesssim \sum_{m''} |\langle f, \varphi_{s_{m''}} \rangle|^2. \quad (5.48)$$

So far we have finished the proof of Lemma 5.7, hence the estimate on each rectangle, which is (5.40).

6 Organizing all rectangles to finish the proof

In this section, we will organize the estimates on all the rectangles together, i.e. to finish the proof of the following estimate

$$\sum_{j,k} \sum_m \int |A^j P_{k+l} f|^2 \mathbb{1}_{R_m} \lesssim 2^{-\mu l} \|f\|_2^2, \quad (6.1)$$

for some $\mu > 0$. To do this, we substitute the estimate (5.40) into the left hand side of the above expression to obtain

$$\sum_{k,j} \sum_m \sum_{\omega \in \mathcal{D}_\theta} \sum_{p,q \in \mathbb{Z}} \frac{2^{-\mu l}}{(|p| + |q| + 1)^M} \sum_{s \in \mathcal{U}_{k+l,\omega}, s \subset \tilde{R}_m^{p,q}} |\langle f, \varphi_s \rangle|^2, \quad (6.2)$$

where we are still using the notation $\theta = k + l - j$. Hence it suffices to prove that for fixed $p, q \in \mathbb{Z}$, we have

$$\sum_{k,j} \sum_m \sum_{\omega \in \mathcal{D}_\theta} \sum_{s \in \mathcal{U}_{k+l,\omega}, s \subset \tilde{R}_m^{p,q}} |\langle f, \varphi_s \rangle|^2 \lesssim (|p| + |q| + 1)^{b_0} \|f\|_2^2, \quad (6.3)$$

where b_0 is the constant in Lemma 4.6.

Proof of the estimate (6.3): We first fix k . For the case $p = q = 0$, for two tiles s' and s'' in the following collection

$$\bigcup_j \bigcup_{\omega \in \mathcal{D}_\theta} \bigcup_m \{s : s \in \mathcal{U}_{k+l,\omega}, s \subset R_m\}, \quad (6.4)$$

we either have

$$\omega_{s'} \cap \omega_{s''} = \emptyset, \quad (6.5)$$

or

$$s' \cap s'' = \emptyset. \quad (6.6)$$

Hence by the (almost) orthogonality, we obtain that

$$\sum_j \sum_m \sum_{\omega \in \mathcal{D}_\theta} \sum_{s \in \mathcal{U}_{k+l,\omega}, s \subset R_m} |\langle f, \varphi_s \rangle|^2 \lesssim \|P_{k+l} f\|_2^2. \quad (6.7)$$

By summing over k , we get the desired estimate (6.3) for the case $p = q = 0$.

For the general $p, q \in \mathbb{Z}$, we no longer have (6.6) due to the simple fact that for two disjoint rectangles (of different dimensions), they might intersect after being translated by (p, q) units separately. Fortunately, Lemma 4.6 says that the intersection caused by translation can only grow polynomially in p and q .

Hence by essentially the same idea as above and by losing a factor of $(|p| + |q| + 1)^{b_0}$, we obtain

$$\sum_j \sum_m \sum_{\omega \in \mathcal{D}_\theta} \sum_{s \in \mathcal{U}_{k+l,\omega}, s \subset \tilde{R}_m^{p,q}} |\langle f, \varphi_s \rangle|^2 \lesssim (|p| + |q| + 1)^{b_0} \|P_{k+l} f\|_2^2. \quad (6.8)$$

Summing over k , we get the estimate (6.3). So far we have finished the proof of (6.1), hence the geometric proof of Theorem 1.1.

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